ISRAEL JOURNAL OF MATHEMATICS, Vol. 74, No. 1, 1991

ON THE COMPLEMENTED SUBSPACES OF X_p

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ABSTRACT

In this paper we prove some results related to the problem of isomorphically classifying the complemented subspaces of X_p. We characterize the complemented subspaces of X_p which are isomorphic to X_p by showing that such a space must contain a canonical complemented subspace isomorphic to X_p. We also give some characterizations of complemented subspaces of X_p isomorphic to $l_p \oplus l_2$.

0. Introduction

Rosenthal [R] introduced the $\mathcal{L}_p$ space X_p in 1971. Among its interesting properties are that it contains and is contained in isomorphs of $l_p \oplus l_2$, but is not isomorphic to a complemented subspace of $l_p \oplus l_2$. These properties have made X_p rather resistant to standard approaches to classifying its complemented subspaces. For example, it was first proved that X_p was primary in [JO2] where the device of simultaneously L_p, L_2 bounded operators was employed to prove a version of the decomposition method for X_p. The problem really is that the l_p and l_2 structures in X_p are mixed in a much different way than they are in $l_p \oplus l_2$ or $(\sum l_2)_p$. Let us also recall that X_p or really the technique for building X_p is the central device used to construct an uncountable number of separable $\mathcal{L}_p$ spaces, [BRS]. Thus a better understanding of X_p is critical for the study of the complemented subspaces of L_p.

In order to state precisely our results we need to introduce some special notation. Throughout this paper $w = (w_n)$ will be a sequence of positive real numbers and $2 < p < \infty$. As usual $X_{p,w}$ is the completion of $\{(a_i) : i \in \mathbf{N},\ a_i \neq 0$ for finitely

†Research supported in part by NSF grant DMS 890237.
Received August 16, 1990

many $i\}$ with the norm $\|(a_i)\| = \max\{|(a_i)|_p, |(a_i)|_{2,w}\}$ where $|(a_i)|_p = [\sum_i |a_i|^p]^{1/p}$ and $|(a_i)|_{2,w} = [\sum_i |a_i|^2 w_i^2]^{1/2}$. The Rosenthal space X_p is $X_{p,w}$ where $w = (w_i)$ is such that for every $\epsilon > 0$, $\sum_{w_i<\epsilon} w_i^{2p/(p-2)} = \infty$. Throughout this paper we will always consider X_p to be the subspace of $(l_p \oplus l_2)_\infty$ spanned by $(\delta_n + w_n\gamma_n)$ where (δ_n) and (γ_n) are the usual unit vector bases of l_p and l_2, respectively. If $E \subset \mathbf{N}$, the symbol $\omega(E)$ will be used to denote $\sum_{n\in E} w_n^{2p/(p-2)}$ which occurs frequently in computations in X_p. We will also need the ratio of 2-norm and p-norm, $|x|_2/|x|_p$, which we will denote by $r(x)$. For a subspace Y of X_p, define $r(Y) = \sup\{r(y) : y \in Y\}$ and $h(Y) = \inf\{r(y) : y \in Y\}$. In defining functionals on X_p it is convenient to use the inner product induced by the norm $|\cdot|_2$. Thus $\langle (x_n), (y_n)\rangle = \sum x_n y_n w_n^2$.

We will use standard Banach space notation and terminology as can be found in [LT]. Here subspace will mean infinite dimensional closed subspace unless otherwise noted. The properties of X_p can be found in [LT, 4d] or in the original paper of Rosenthal [R].

1. Complemented subspaces of X_p which contain X_p complemented

In this section we will show that a complemented subspace of X_p which contains a complemented subspace isomorphic to X_p contains a canonical complemented copy of X_p. In [R] Rosenthal showed that there were nice block bases of the usual basis of X_p with complemented closed linear span. The point of his construction was to make sure that the coordinate functionals of the block basis could be chosen to be bounded in both the p and 2 norms. Explicitly, if for each $j \in \mathbf{N}$,

$$y_j = \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} e_n$$

where (e_n) is the natural basis for X_p and (e_n^*) is the corresponding sequence of biorthogonal functionals, then

$$|y_j|_2 = \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{1/2} \quad \text{and} \quad |y_j|_p = \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{1/p}.$$

Let

$$y_j^*(x) = |y_j|_2^{-2} \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x) = |y_j|_2^{-2}\langle y_j, x\rangle.$$

By applying Hölder's inequality first in l_2 and then in $l_{p/2}$ we see that

$$\begin{aligned}|y_j^*(x)| &= |y_j|_2^{-2}\left|\sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x)\right| \\ &\le |y_j|_2^{-2}\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{4/(p-2)} w_n^2\right]^{1/2}\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2\right]^{1/2} \\ &\le |y_j|_2^{-2}\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{1/2} \min\left\{\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2\right]^{1/2}, \right. \\ &\qquad \left.\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{(p-2)/2p}\left[\sum_{n=k_j+1}^{k_{j+1}} |e_n^*(x)|^p\right]^{1/p}\right\} \\ &= \min\{|x_{|[k_j+1,k_{j+1}]}|_2 |y_j|_2^{-1}, |x_{|[k_j+1,k_{j+1}]}|_p |y_j|_p^{-1}\}.\end{aligned}$$

Thus

$$|y_j^*(x) y_j|_2 \le |x_{|[k_j+1,k_{j+1}]}|_2 \quad \text{and} \quad |y_j^*(x) y_j|_p \le |x_{|[k_j+1,k_{j+1}]}|_p.$$

The important point is that this computation works because

$$\begin{aligned}|y_j|_2 &= \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{(p-2)/2p}\left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{1/p} \\ &= \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{(p-2)/2p} |y_j|_p.\end{aligned}$$

REMARK 1.1. The choice of coefficients for y_j has the following geometric motivation. These coefficients give the maximum for the ratio $|\cdot|_2/|\cdot|_p$ for elements supported on $[k_j+1, k_{j+1}]$. Thus if x is supported on $[k_j+1, k_{j+1}]$ and $x \in B_{l_p}$ (the unit ball of l_p) then $(|y_j|_p/|y_j|_2)x \in B_{|\cdot|_2}$ and there is exactly one x so that the multiple has 2-norm one.

If we replace y_j by any z_j with the same support which satisfies

$$c|z_j|_2 \ge \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}\right]^{(p-2)/2p} |z_j|_p = \omega([k_j+1, k_{j+1}])^{(p-2)/2p}|z_j|_p$$

and define

$$z_j^*(x) = |z_j|_2^{-2}\langle z_j, x\rangle = |z_j|_2^{-2}\sum_{n=k_j+1}^{k_{j+1}} e_n^*(z_j) e_n^*(x) w_n^2,$$

applying Hölder's inequality as above shows that

$$|z_j^*(x)z_j|_2 \le |x_{|[k_j+1,k_{j+1}]}|_2 \quad\text{and}\quad |z_j^*(x)z_j|_p \le c|x_{|[k_j+1,k_{j+1}]}|_p.$$

Combining this observation with the isomorphic classification of subspaces spanned by block basic sequences, we arrive at a prototype for a complemented subspace of X_p isomorphic to $X_{p,w'}$.

PROPOSITION 1.2. *If* (z_j) *is a normalized block basis of the natural basis of* X_p, (E_j) *is a sequence of disjoint subsets of* $\mathbf{N}$ *and there are positive constants* c *and* δ *such that for all* j

(a) $|z_{j|E_j}|_2 \ge \delta|z_j|_2$,

(b) $c|z_{j|E_j}|_2 \ge \omega(E_j)^{(p-2)/2p}$.

Then $[z_j]$ *is a* $\max\{\delta^{-1},c\}$ *complemented subspace of* X_p *isomorphic to* $X_{p,w'}$, *where* $w_j' = \omega(E_j)^{(p-2)/2p}$.

PROOF. A block basis of the natural basis of X_p spans a subspace isomorphic to $X_{p,w''}$ where $w_j'' = |z_j|_2/|z_j|_p = r(z_j)$. By Hölder's inequality and (b),

$$|z_j|_p\omega(E_j)^{(p-2)/2p} \ge |z_{j|E_j}|_p\omega(E_j)^{(p-2)/2p} \ge |z_{j|E_j}|_2 \ge \delta|z_j|_2.$$

Hence

$$\delta^{-1}\omega(E_j)^{(p-2)/2p} \ge r(z_j) \ge c^{-1}\omega(E_j)^{(p-2)/2p}$$

and thus $X_{p,w'}$ is isomorphic to $X_{p,w''}$. Define a projection onto $[z_j : j \in \mathbf{N}]$ by

$$\begin{aligned} Px &= \sum_{j=1}^{\infty} z_j^*(x)z_j \\ &= \sum_{j=1}^{\infty} |z_{j|E_j}|_2^{-2}\langle z_{j|E_j}, x\rangle z_j \\ &= \sum_{j=1}^{\infty} |z_{j|E_j}|_2^{-2}\left[\sum_{n\in E_j} e_n^*(z_j)e_n^*(x)w_n^2\right] z_j. \end{aligned}$$

Clearly P is the required operator if it is bounded. The computations above using (b) in the form

$$(c|z_{j|E_j}|_p)|z_{j|E_j}|_2 \ge \omega(E_j)^{(p-2)/2p}|z_{j|E_j}|_p$$

show that

$$\begin{aligned}\|Px\| &\le \max\left\{\left[\sum_{j=1}^{\infty} |z_j^*(x)z_j|_2^2\right]^{1/2}, \left[\sum_{j=1}^{\infty} |z_j^*(x)z_j|_p^p\right]^{1/p}\right\} \\ &= \max\left\{\left[\sum_{j=1}^{\infty} |z_j^*(x)z_{j|E_j}|_2^2 |z_j|_2^2 / |z_{j|E_j}|_2^2\right]^{1/2},\right. \\ &\qquad \left.\left[\sum_{j=1}^{\infty} |z_j^*(x)z_{j|E_j}|_p^p |z_j|_p^p / |z_{j|E_j}|_p^p\right]^{1/p}\right\} \\ &\le \max\left\{\left[\sum_{j=1}^{\infty} |x_{|E_j}|_2^2 \delta^{-2}\right]^{1/2}, \left[\sum_{j=1}^{\infty} |x_{|E_j}|_p^p c^p\right]^{1/p}\right\} \\ &\le \max\{\delta^{-1}, c\}\|x\|. \end{aligned}$$

■

Next we will prove our characterization of the complemented subspaces of X_p which contain X_p complemented and thus are isomorphic to X_p, [JO2].

THEOREM 1.3. *Suppose that X is a complemented subspace of X_p. Then the following are equivalent*:

(1) *X contains a complemented subspace isomorphic to X_p.*

(2) *There exist positive constants c and δ such that for every $\epsilon > 0$ there is an ϵ', $0 < \epsilon' < \epsilon$, such that for every $N \in \mathbf{N}$ there is an $x \in X$, $\|x\| = 1$ and a finite set $E \subset \{N, N+1, \dots\}$ such that*

(a) $\|x_{|[1,N]}\| < N^{-1}$,

(b) $|x_{|E}|_2 \ge \delta |x|_2$,

(c) $\epsilon \ge c|x_{|E}|_2 \ge \omega(E)^{(p-2)/2p} \ge \epsilon'$.

PROOF. Suppose that (2) is satisfied. Let $\epsilon_k = k^{-1}$. By induction we may choose for each k a sequence $(x_{k,j})$ of norm one elements of X which are a perturbation of a block basis of the basis of X_p satisfying (b) and (c) (for ϵ_k' and $E_{k,j}$). Clearly we may assume that $E_{k,j} \cap E_{k,m} = \emptyset$ for $j \ne m$. By a simple diagonalization argument we can find sets $\mathfrak{F}_k \subset \mathbf{N}$ such that $(x_{k,j})_{k=1, j \in \mathfrak{F}_k}^{\infty}$ is equivalent to a block basis of X_p, the sets $E_{k,j}$, $k \in \mathbf{N}, j \in \mathfrak{F}_k$ are disjoint, and $(\epsilon_k')^{2p/(p-2)} \operatorname{card} \mathfrak{F}_k \ge 1$ for each k. It now follows from Proposition 1.2 and standard perturbation arguments that $Y = [x_{k,j} : k \in N, j \in \mathfrak{F}_k]$ is isomorphic to X_p and that Y is complemented in X_p.

For the converse we will actually show that if we take as our isomorph of X_p the special representation $X_{p,w'}$, where w' is actually a doubly indexed sequence

$w' = (w_{k,j})$, where $w_{k,j} = w_k$ for all j, $\lim w_k = 0$, and $\sum_{k=1}^{\infty} w_k^{2p/(p-2)} = \infty$, then the images of a subsequence of the basis satisfy the properties in (2). Thus we suppose that Y is a complemented subspace of X and that T is an isomorphism of $X_{p,w'}$ onto Y. By passing to a subsequence of the basis of $X_{p,w'}$ and using a standard perturbation argument, we may assume that Y is the span of a block of the basis of the containing X_p. Let (y_i) be the normalized basis of Y and let F_i be the support of y_i relative to the basis of X_p. Let y_i^* denote the biorthogonal functional to y_i. Because Y is complemented in X_p, we may assume that each y_i^* is defined on X_p and $\sup \| y_i^* \| \leq \| T^{-1} \| \, \| Q \|$ where Q is the projection onto Y. Because Y is reflexive, $y_i^*(e_j) \to 0$ as $i \to \infty$ for each j where e_j denotes the jth basis vector of X_p. Thus we may assume, by passing to a subsequence, a perturbation argument and perhaps enlarging the sets F_i slightly that $y_i^*(x) \neq 0$ only if $x_{|F_i} \neq 0$. In other words $y_i^*(x) = y_i^*(x_{|F_i})$. Also it follows from this that the projection Q onto Y is given by $Qx = \sum_{i=1}^{\infty} y_i^*(x) y_i$.

Fix i and let $E_i = \{ j \in F_i : | y_i(j) | \geq \rho w_j^{2/(p-2)} | y_i |_2^{-2/(p-2)} \}$ and assume that (b) is satisfied for y_i and E_i.

$$\begin{aligned}\omega(E_i) = \sum_{j \in E_i} w_j^{2p/(p-2)} &\leq \rho^{-2} \sum_{j \in E_i} | y_i(j) |^2 | y_i |_2^{4/(p-2)} w_j^2 \\ &= \rho^{-2} | y_{i|E_i} |_2^2 | y_i |_2^{4/(p-2)} \\ &\leq \rho^{-2} \delta^{-4/(p-2)} | y_{i|E_i} |_2^{2p/(p-2)}.\end{aligned}$$

Thus if ρ is independent of i, condition (b) will imply the middle inequality in condition (c) (with $c = \rho^{-(p-2)/p} \delta^{-2/p}$). Also observe that because

$$| y_i |_p \omega(E_i)^{(p-2)/2p} \geq | y_{i|E_i} |_p \omega(E_i)^{(p-2)/2p} \geq | y_{i|E_i} |_2 \geq \delta | y_i |_2$$

the third inequality in (c) will be satisfied if $| y_i |_2$ is bounded away from zero. Hence it is sufficient to show that for some $\rho > 0$ there is a δ, $0 < \delta < 1$, such that if $\mathcal{E}_\delta = \{ i : | y_{i|E_i} |_2 \geq \delta | y_i |_2 \}$ then for every $\epsilon_1 > 0$ there is an $\epsilon_2 > 0$ such that $\epsilon_1 \geq | y_i |_2 \geq \epsilon_2$ for infinitely many i in $\mathcal{E}_\delta$. Then (c) will be satisfied with $c\epsilon_1 = \epsilon$ and $\epsilon' = \delta \epsilon_2$.

Note that $\sum_{j \notin E_i} | y_i(j) |^p \leq \sum_{j \notin E_i} | y_i(j) |^2 \rho^{p-2} w_j^2 | y_i |_2^{-2} \leq \rho^{p-2}$. Hence $\| y_{i|E_i} \| \geq [1 - \rho^{p-2}]^{1/p}$ and $| y_{i|F_i \setminus E_i} |_p \leq \rho^{1-2/p}$. Thus if ρ is small, $[y_{i|E_i^c} : i \notin \mathcal{E}_\delta]$ is not better than $\rho^{-1+2/p}$ equivalent to $[y_i : i \notin \mathcal{E}_\delta]$.

For each K define $\mathcal{M}_K = \{ i : | y_i^*(y_{i|E_i}) | \leq K | y_{i|E_i} |_2 / | y_i |_2 \}$. Observe that because $\| y_i^* \| \leq \| T^{-1} \| \, \| Q \|$,

$$|y_i^*(y_{i|F_i\setminus E_i})| \le \|T^{-1}\|\,\|Q\|\max\{|y_{i|F_i\setminus E_i}|_2, |y_{i|F_i\setminus E_i}|_p\}$$
$$\le \|T^{-1}\|\,\|Q\|\max\{|y_i|_2, \rho^{1-2/p}\}.$$

Thus if we consider only those y_i with $|y_i|_2 \le \rho^{1-2/p}$ we have that

$$|y_i^*(y_{i|E_i})| \ge 1 - \|T^{-1}\|\,\|Q\|\rho^{1-2/p}.$$

Under our assumption on the sequence (y_i), the span of such y_i is still isomorphic to $X_{p,w}$. From now on we will assume that $\rho^{1-2/p} \le (\|T^{-1}\|\,\|Q\|2)^{-1}$ and thus that $|y_i^*(y_{i|E_i})| \ge \frac{1}{2}$ for all i. In this way we can work with $\mathcal{M}_K$ instead of $\mathcal{E}_\delta$ since, for such i, if $i \in \mathcal{M}_K$ then $i \in \mathcal{E}_{1/2K}$.

Let us now see how the projection onto $[y_i]$ acts on the span of $[y_{i|E_i}]$. Our assumptions on the y_i's imply that $Q\sum_{i=1}^\infty a_i y_{i|E_i} = \sum_{i=1}^\infty a_i y_i^*(y_{i|E_i})y_i$. Hence

$$\left|Q\sum_{i=1}^\infty a_i y_{i|E_i}\right|_2 = \left|\sum_{i=1}^\infty a_i y_i^*(y_{i|E_i})y_i\right|_2$$
$$\ge K\left[\sum_{i\notin\mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2 |y_i|_2^{-2} |y_i|_2^2\right]^{1/2}$$
$$= K\left[\sum_{i\notin\mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2\right]^{1/2}.$$

If $a_i = |y_{i|E_i}|_2^{2/(p-2)}$, for $i \notin \mathcal{M}_K$ and $i \le N$, and 0 else, then

$$\left\|\sum_{i=1}^N a_i y_{i|E_i}\right\| = \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |a_i|^p |y_{i|E_i}|_p^p\right]^{1/p}, \left[\sum_{i\notin\mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2\right]^{1/2}\right\}$$
$$\le \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} |y_i|_p^p\right]^{1/p}, \left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2}\right\}$$
$$\le \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/p}, \left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2}\right\}$$
$$= \sum_{i\notin\mathcal{M}_K} [|y_{i|E_i}|_2^{2p/(p-2)}]^{1/2}, \qquad \text{if } \sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \ge 1.$$

This implies that

$$\|Q\|\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2} \ge K\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2},$$

if $\sum_{i\notin \mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \ge 1$. Therefore, if $K \ge \|Q\|$,

$$\sum_{i\notin \mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \le 1.$$

Because $|y_{i|E_i}|_p \ge [1-\rho^{p-2}]^{1/p}$ (we are assuming that $\rho < 1$) this implies that $[y_{i|E_i}\|y_{i|E_i}\|^{-1} : i \notin \mathcal{M}_K]$ is equivalent to the basis of l_p. Therefore for any ϵ' small enough only finitely many of the y_i's with $|y_i|_2 \ge \epsilon'$ have index not in $\mathcal{M}_K$. Indeed, if this were not the case, then there would be a subsequence of (y_i), say $(y_i)_{i\in M}$, such that $M \subset \mathcal{M}_K^c$ and $\epsilon > 0, |y_i|_2 \ge \epsilon$ for all $i \in M$. However, this would imply that Q is an isomorphism from $[y_{i|E_i} : i \in M]$, which is isomorphic to l_p, onto $[y_i : i \in M]$, which is isomorphic to l_2.

It follows that $(y_i)_{i\in\mathcal{M}_K}$ is equivalent to the basis of $X_{p,w'}$, where $w' = (w_i')$ and for each $\epsilon > 0$ there is a $\epsilon' > 0$ such that $\epsilon > w_i' \ge \epsilon'$, for infinitely many i. Note that because $|y_i^*(y_{i|E_i})| \ge \frac{1}{2}$, for any $i \in \mathcal{M}_K$, $|y_{i|E_i}|_2 \ge \delta |y_i|_2$, where $\delta = 1/2K$, i.e., $i \in \mathcal{E}_\delta$. ■

Remark 1.4. If X is not isomorphic to X_p we can use part of the proof that (1) implies (2) to get a natural way of splitting vectors in X into a piece with large ratio and a piece with small ratio. Indeed suppose X is a complemented subspace of X_p and that P is the projection onto X. By [JJ] or [JO2] we may assume that P is bounded in the norm $|\cdot|_2$ as well. Suppose that (2) fails for $\delta < |P|_2^{-1}$, c and ϵ. Choose positive constants ϵ', ρ, α, and β such that

$$\epsilon' < \min\{\epsilon, \delta\alpha\},$$

$$\beta < \min\{(1-\delta|P|_2)/(\|P\|), \epsilon/c\},$$

$$\rho \le \min\{c^{-p/(p-2)}\delta^{2/(p-2)}, \beta^{p/(p-2)}\},$$

$$\beta > \alpha \ge \max\{\beta\delta|P|_2/(1-\beta\|P\|), \beta^2\|P\|/(1-\delta|P|_2).$$

Let N be an integer so that (a), (b), and (c) of (2) fail for ϵ' and N and suppose that $x \in X$, $x|_{[1,N]} = 0$, $\alpha < r(x) < \beta$ and $\|x\| = 1$. Let $E_x = \{j : |x(j)| \ge \rho w_j^{2/(p-2)} |x|_2^{-2/(p-2)}\}$. As in the proof above, the choice of ρ guarantees that the middle inequality in (2)(c) is satisfied by $x_{|E_x}$. Because $r(x) < \beta \le \epsilon/c$, the first inequality in (2)(c) is also satisfied. Finally, if $|x_{|E_x}|_2 \ge \delta|x|_2$, $\omega(E_x)^{(p-2)/2p} \ge |x_{|E_x}|_2 \ge \delta|x|_2 \ge \delta\alpha \ge \epsilon'$ and thus all of the inequalities in (c) are satisfied. The failure of (2) then implies that $|x_{|E_x}|_2 < \delta|x|_2$.

Let $y = P(x_{|E_x})$ and $z = x - y = P(x_{|E_x^c})$. We claim that $r(y) \le \alpha$ and $r(z) \ge \beta$. Indeed,

$$|y|_2 \leq |P|_2 |x_{|E_x}|_2 < |P|_2 \delta |x|_2 \leq |P|_2 \delta\beta$$
$$\leq \alpha(1 - \beta\|P\|) \leq \alpha(|x|_p - \|P\| \, \|x_{|E_x^c}\|) \leq \alpha |y|_p$$

since $|x_{|E_x^c}|_p \leq \rho^{(p-2)/p} \leq \beta$ and $\|x\|_2 \leq \beta$, and

$$|z|_2 \geq (1 - \delta|P|_2)\alpha \geq (1 - \delta|P|_2)\beta^2\|P\|/(1 - \delta|P|_2) \geq \beta\|P\| \, \|x_{|E_x^c}\| \geq \beta|z|_p.$$

Thus any $x \in X$ with support in $\{N+1, N+2, \dots\}$ can be split into an element with ratio greater than β and one with ratio smaller than α. If this could be accomplished in a linear fashion it would follow that X is then isomorphic to a complemented subspace of $l_p \oplus l_2$.

2. Complemented subspaces of X_p which are isomorphic to $l_p \oplus l_2$

In this section we look at some ways of discriminating between complemented subspaces of X_p which are isomorphic to complemented subspaces of $l_p \oplus l_2$ and those isomorphic to X_p. First we will examine how the conditions in Theorem 1.3 fail if X is isomorphic to $l_p \oplus l_2$. Below P_n denotes the basis projection onto the span of the first n elements of the basis of X_p.

Proposition 2.1. *Suppose that Z, X, U, and W are subspaces of X_p such that $Z \subset X = U \oplus W$, U is isomorphic to l_2 and W is isomorphic to l_p. Suppose that Z has a normalized K unconditional basis (z_n). Let $\beta = \lim_{n\to\infty} r(z_n)$ and $\beta' = \lim_{n\to\infty} \inf\{b :$ for every $\epsilon > 0$ there exists $u \in U$ such that $\|P_n u\| < \epsilon$ and $r(u) \leq b\}$. If $\beta > 0$, $\beta' \leq 1$, and P is a projection from X onto Z then $\|P\| \geq \beta'/K\beta$.*

Proof. Let $z_n = u_n + w_n$ where $u_n \in U$ and $w_n \in W$. By passing to subsequences and a standard perturbation argument we may assume that (u_n) and (w_n) are block bases of the basis of X_p. (It could happen that $\|w_n\| \to 0$, but then $\beta' \leq \beta$.) Moreover we may assume that the projection P composed with the corresponding basis projection Q acts disjointly with respect to the subsequence $(z_n)_{n\in M}$, i.e., QP is a projection onto $[z_n : n \in M]$ and $QPu_n = \tau_n z_n$ and $QPw_n = (1 - \tau_n)z_n$. Because (w_n) is equivalent to the usual unit vector basis of l_p, $p > 2$, $\beta > 0$, and (z_n) is equivalent to the usual unit vector basis of l_2, it follows that $\tau_n \to 1$.

Because W is isomorphic to l_p, $|w_n|_2 \to 0$ and thus $|u_n|_2 - |z_n|_2 \to 0$. Therefore

$$\limsup\|u_n\| = \limsup \max\{|u_n|_2, |u_n|_p\} \leq \limsup \max\{\beta, |u_n|_2/r(u_n)\}$$
$$\leq \max\{\beta, \beta/\beta'\} = \beta/\beta'.$$

Consequently $K\|P\|\beta/\beta' \geq \limsup\|Q\| \, \|P\| \, \|u_n\| \geq 1.$ ∎

COROLLARY 2.2. *Suppose that X, U, and W are subspaces of X_p which satisfy the hypotheses of Proposition* 2.1 *and X is complemented in X_p with projection P. Then for any c and δ and $\epsilon < \beta' c\delta/\max\{c,\delta^{-1}\}$, there is no ϵ', $0 < \epsilon' < \epsilon$, such that for every $N \in \mathbf{N}$ there is an $x \in X$, $\|x\| = 1$ and a finite set $E \subset \{N, N+1, \dots\}$ such that*

(a) $\|x_{|[1,N]}\| < N^{-1}$,

(b) $|x_{|E}|_2 \geq \delta |x|_2$,

(c) $\epsilon \geq c|x_{|E}|_2 \geq \omega(E)^{(p-2)/2p} \geq \epsilon'$.

PROOF. Suppose ϵ' exists for some c, δ, and ϵ. Then there is a sequence (z_n) of norm one vectors in X which is a perturbation of a block basis of the basis of X_p and disjoint sets (E_n) such that for all n

(b) $|z_{n|E_n}|_2 \geq \delta |z_n|_2$;

(c) $\epsilon \geq c|z_{n|E_n}|_2 \geq \omega(E_n)^{(p-2)/2p} \geq \epsilon'$.

Then $[z_n : n \in \mathbf{N}]$ is complemented in X_p by a projection of norm at most $\max\{c,\delta^{-1}\}$ and $r(z_n) \leq \epsilon/c\delta$. Thus by the previous proposition $\|P\| \geq \beta' c\delta/\epsilon$ and hence

$$\epsilon \geq \beta' c\delta/\max\{c,\delta^{-1}\}. \qquad \blacksquare$$

We now turn our attention to the classification of the complemented subspaces of X_p. It was shown in [JO2] that if a complemented subspace of X_p has an unconditional basis then it is isomorphic to $l_p, l_2, l_p \oplus l_2$, or X_p. In [AC] the same conclusion was established if X has a "p,2 F.D.D." Thus it seems likely that same result holds without the additional assumptions. We will next look at some well-known results but recast in terms of the ratio of the 2-norm and p-norm.

To begin, let us recall that results of Kadec and Pełczyński [KP] give a natural criterion for isomorphs of l_2 contained in X_p, $p > 2$, namely, a subspace X of X_p is isomorphic to l_2 if and only if there is a constant $C > 0$ such that $r(x) = |x|_2/|x|_p \geq C$ for all $x \in X$, i.e., $h(X) \geq C$. It follows from [JO1] that if a complemented subspace of X_p does not contain l_2 then it is isomorphic to l_p. A standard gliding hump argument yields the following criterion. (Below, Q_N denotes the projection onto the span of the basis vectors of X_p with index greater than N.)

PROPOSITION 2.3. *A complemented subspace X of X_p is isomorphic to l_p if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that if $x \in Q_N X$ then $r(x) < \epsilon$.*

Theorem 1.3 gives a criterion for identifying complemented subspaces isomorphic to X_p, however it seems to be rather difficult to formulate useful conditions which identify complemented subspaces isomorphic to $l_2 \oplus l_p$ or a complemented subspace of it. Here are some attempts at such criteria.

PROPOSITION 2.4. *Let X be a complemented subspace of X_p and suppose that Z is a subspace of X and ϵ, β, and β' are positive constants with $\epsilon \le 1$ such that*

(a) *for all $z \in Z$, $r(z) \ge \beta'$,*

(b) *if $x \in X$ and $r(x) > \beta$ then there exists $z \in Z$ such that $|x - z|_2 < \epsilon |x|_2$.*

Then X is isomorphic to a complemented subspace of $l_p \oplus l_2$ and conversely.

PROOF. Condition (a) implies that Z is isomorphic to l_2. Let Y be the kernel of the orthogonal projection from X onto Z. If $y \in Y$ and $r(y) > \beta$ then by (b) there exists a $z \in Z$ such that $|y - z|_2 < \epsilon |y|_2$. But y is orthogonal to z so we have that $|y|_2 \le [|y|_2^2 + |z|_2^2]^{1/2} = |y - z|_2 < \epsilon |y|_2$, an impossibility. Therefore $r(y) \le \beta$ for all $y \in Y$ and Y is isomorphic to l_p.

If X is isomorphic to l_2 or l_p the converse follows easily from our earlier observations. Thus by the results of Edelstein and Wojtaszczyk [EW] we may assume that X is isomorphic to $l_p \oplus l_2$ and let U and W be the complementary subspaces with U isomorphic to l_2 and W isomorphic to l_p. Because U is isomorphic to l_2 there is a constant $1 \ge \beta' > 0$ such that for all $u \in U$, $r(u) \ge \beta'$. We may also assume that W is the kernel of the orthogonal projection Q onto U.

Then for any $x \in X$,

$$\begin{aligned} |x - Qx|_2 &\le r(W)\|x - Qx\| \le r(W)(1 + \|Q\|)\|x\| \\ &\le r(W)(1 + 1/\beta')\|x\|_2 \max\{1, 1/r(x)\}. \end{aligned}$$

Because W is isomorphic to l_p it has a basis, let R_n denote $I - Q$ composed with the basis projection onto the span of the first n elements of the basis of W. Let $K = \sup\|R_n\|$. Choose n so large that if $Y = (I - R_n)W$, $r(Y) < (2(1 + 1/\beta') \times (1 + K))^{-1}$. Then if $x \in Y + Z$ the above computation shows that (b) is satisfied with $\epsilon = \frac{1}{2}$. Because R_n is finite rank there exists a $\beta > 1$ such that if $r(x) > \beta$ then $\|R_n x\| < \|x\|/4(1 + \|Q\|)$. Now if $r(x) > \beta$,

$$\begin{aligned} |x - Qx|_2 &\le (1 + \|Q\|)\|R_n x\| + |(I - R_n)x - Q(I - R_n)x|_2 \\ &\le (1 + \|Q\|)\|x\|/4(1 + 1/\beta') + r(Y)\|(I - R_n)x - Q(I - R_n)x\| \\ &\le |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + \|Q\|)\|I - R_n\|\,\|x\| \\ &\le |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + 1/\beta')(1 + K)|x|_2 \max\{1, 1/r(x)\} \\ &\le (3/4)|x|_2. \end{aligned}$$

■

REMARK 2.5. Condition (b) may be replaced by

(b′) if $x \in X$ and $r(x) > \beta''$ then there exists $z \in Z$ such that $|x - z|_2 < \epsilon \|x\|$.

To see this note that if (b′) holds then (b) holds with $\beta = \max\{\beta'', \epsilon\}$ and $\epsilon = 1$.

To get a similar theorem but with the hypothesis on the l_p part we seem to need to assume the existence of a projection.

PROPOSITION 2.6. *Let X be a complemented subspace of X_p and suppose that Y is the range of a projection P on X and ϵ, α, and α' are positive constants with $\epsilon \leq \|I - P\|^{-1}$ such that*

(a) *for all $y \in Y$, $r(y) \leq \alpha'$,*

(b) *if $x \in X$ and $r(x) < \alpha$ then there exists $y \in Y$ such that $\|x - y\| < \epsilon\|x\|$.*

Then X is isomorphic to a complemented subspace of $l_p \oplus l_2$ and conversely.

PROOF. Condition (a) implies that Y is isomorphic to l_p. Let Z be the kernel of the projection P from X onto Y. If $z \in Z$ and $r(z) < \alpha$ then by (b) there exists a $y \in Y$ such that $\|z - y\| < \epsilon\|z\|$. But $Pz = 0$ and $Py = y$ so we have that $\|z\| = \|(I - P)(z - y)\| < \|I - P\|\epsilon\|z\| \leq \|z\|$, an impossibility. Therefore $r(z) \geq \alpha$ for all $z \in Z$ and Z is isomorphic to l_2.

As in the proof of Proposition 2.6 the converse easily reduces to the case that X is isomorphic to $l_2 \oplus l_p$. So we again let U and W be the complementary subspaces with U isomorphic to l_2 and W isomorphic to l_p and let β' be a constant such that $1 \geq \beta' > 0$ and, for all $u \in U$, $r(u) \geq \beta'$. As before we will assume that W is the kernel of the orthogonal projection Q onto U.

Then for any $x \in X$,

$$\|Qx\| = \max\{|Qx|_2, |Qx|_p\} \leq \max\{|Qx|_2, |Qx|_2/\beta'\} \leq \langle Qx, x\rangle^{1/2}/\beta'$$
$$\leq [|Qx|_2 r(x)\|x\|]^{1/2}/\beta' \leq \|Q\|^{1/2} r(x)^{1/2}\|x\|/\beta'.$$

Thus if $r(x) < \alpha = \beta'^2/\|Q\|^3$, $\|x - (I - Q)x\| = \|Qx\| < \|x\|/\|I - (I - Q)\|$. Because Y is isomorphic to l_p there is some α' such that $r(y) \leq \alpha'$ for all $y \in Y$. ■

REMARK 2.7. Propositions 2.3, 2.4, and 2.6 do not really use the structure of X_p and thus can be restated for complemented subspaces of L_p.

Proposition 2.6 should be compared to the following result for X_p itself.

PROPOSITION 2.8. *There does not exist a subspace Y of X_p and positive constants ϵ and α, $\epsilon < 1$ such that*

(a) *$r(Y) < \infty$,*

(b) *if $x \in X_p$ and $r(x) < \alpha$ then there exists a $y \in Y$ with $\|x - y\| < \epsilon\|x\|$.*

PROOF. Suppose such a subspace exists. Then there is a normalized block basic sequence (x_n) of the X_p basis such that $\alpha > r(x_n) > \alpha/2$ for all n and such that $[x : n \in \mathbf{N}]$ is norm one complemented in X_p with projection P. By (b) for each n there is an element y_n of Y such that $\|x_n - y_n\| < \epsilon\|x_n\|$. Because P is norm 1,

$\|Py_n - x_n\| < \epsilon < 1$. Hence $\|Py_n\| > 1 - \epsilon$. By passing to a subsequence we may assume that (y_n) is equivalent to the usual unit vector basis of l_p and that (Py_n) is equivalent to a block basic sequence in $[x_n : n \in \mathbf{N}]$. But (x_n) is equivalent to the unit vector basis of l_2 and hence so is (Py_n). Because $p > 2$ this is a contradiction. ■

REMARK 2.9. The above proposition fails if $\epsilon = 1$. In this case the span of a perturbation of a natural basic sequence equivalent to the basis for l_p may be used for Y.

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